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An energy criterion for the linear stability of conservative flows

By P. A. DAVIDSON

Engineering Department, University of Cambridge, Trumpington Street, Cambridge CB2 1PZ, UK

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We investigate the linear stability of inviscid flows which are subject to a conservative body force. This includes a broad range of familiar conservative systems, such as ideal MHD, natural convection, flows driven by electrostatic forces and axisymmetric, swirling, recirculating flow. We provide a simple, unified, linear stability criterion valid for any conservative system. In particular, we establish a *principle of maximum action* of the form

$$e = \frac{1}{2}\int \dot{\boldsymbol{\eta}}^2\,\mathrm{d}V - \mathrm{d}^2 L(\boldsymbol{\eta}) = \text{constant}, \quad \mathrm{d}^1 L(\boldsymbol{\eta}) = 0,$$

$$\mathrm{d}^2 L(\boldsymbol{\eta}) = \mathrm{d}^2 T(\boldsymbol{\eta}) - \mathrm{d}^2 V(\boldsymbol{\eta}),$$

where $\boldsymbol{\eta}$ is the Lagrangian displacement, e is a measure of the disturbance energy, T and V are the kinetic and potential energies, and L is the Lagrangian. Here d represents a variation of the type normally associated with Hamilton's principle, in which the particle trajectories are perturbed in such a way that the time of flight for each particle remains the same. (In practice this may be achieved by advecting the streamlines of the base flow in a frozen-in manner.) A simple test for stability is that e is positive definite and this is achieved if $L(\boldsymbol{\eta})$ is a maximum at equilibrium. This captures many familiar criteria, such as Rayleigh's circulation criterion, the Rayleigh–Taylor criterion for stratified fluids, Bernstein's principle for magnetostatics, Frieman & Rotenberg's stability test for ideal MHD equilibria, and Arnold's variational principle applied to Euler flows and to ideal MHD. There are three advantages to our test: (i) $\mathrm{d}^2 T(\boldsymbol{\eta})$ has a particularly simple quadratic form so the test is easy to apply; (ii) the test is universal and applies to any conservative system; and (iii) unlike other energy principles, such as the energy-Casimir method or the Kelvin–Arnold variational principle, there is no need to identify all of the integral invariants of the flow as a precursor to performing the stability analysis. We end by looking at the particular case of MHD equilibria. Here we note that when $\boldsymbol{u}$ and $\boldsymbol{B}$ are co-linear there exists a broad range of stable steady flows. Moreover, their stability may be assessed by examining the stability of an equivalent magnetostatic equilibrium. When $\boldsymbol{u}$ and $\boldsymbol{B}$ are non-parallel, however, the flow invariably violates the energy criterion and so could, but need not, be unstable. In such cases we identify one mode in which the Lagrangian displacement grows linearly in time. This is reminiscent of the short-wavelength instability of non-Beltrami Euler flows.

1. Introduction

1.1. *Our model system*

We are interested in the linear stability of non-dissipative flows. In particular, we shall develop a stability criterion which may be used to examine the stability of steady solutions of any incompressible, conservative system. However, it is convenient to have a model system to illustrate the ideas. We shall choose

$$\frac{\partial \boldsymbol{\Omega}}{\partial t} = \nabla \times (\boldsymbol{u} \times \boldsymbol{\Omega}) - \nabla \times (\boldsymbol{H} \times \boldsymbol{J}) + \nabla \times (f \nabla \Phi), \tag{1.1}$$

$$\frac{\partial \boldsymbol{H}}{\partial t} = \nabla \times (\boldsymbol{u} \times \boldsymbol{H}), \quad \frac{\mathrm{D}\Phi}{\mathrm{D}t} = 0, \tag{1.2}$$

with the boundary conditions: $\boldsymbol{u} \cdot \boldsymbol{n} = \boldsymbol{H} \cdot \boldsymbol{n} = \Phi = 0$ on the boundary. Here $\boldsymbol{u}$ is the velocity field, $\boldsymbol{\Omega}$ is the vorticity, $\boldsymbol{H}$ is some *frozen-in* vector field which could, though need not, be a magnetic field, $\boldsymbol{B}$, and Φ is any materially conserved scalar field.† The auxiliary field f is taken as a prescribed function of position $f = f(\boldsymbol{x})$, and $\boldsymbol{J}$ is defined through $\boldsymbol{J} = \nabla \times \boldsymbol{H}$. This encompasses a broad range of familiar conservative flows, including Euler flows, ideal MHD flow, motion driven by buoyancy, flows driven by electrostatic forces, and poloidal motion driven by the centrifugal force associated with some frozen-in distribution of angular momentum, i.e. swirling, recirculating flow. We shall develop a single, universal stability criterion for these flows. Note that (1.1) can also be written as

$$\frac{\partial \boldsymbol{u}}{\partial t} = \boldsymbol{u} \times \boldsymbol{\Omega} - \nabla C + f \nabla \Phi - \boldsymbol{H} \times \boldsymbol{J}, \tag{1.3}$$

where C is Bernoulli's function.

1.2. *A general stability criterion based on the Lagrangian*

We shall show that flows governed by (1.1), or (1.3), and (1.2) have the property that, for virtually all (physically realizable) types of small-amplitude disturbances,

$$e = \frac{1}{2} \int \dot{\boldsymbol{\eta}}^2 \,\mathrm{d}V - \mathrm{d}^2 L = \text{constant}, \quad \mathrm{d}^1 L = 0. \tag{1.4}$$

Here $L = T - V$ is the Lagrangian, $\boldsymbol{\eta}$ is (closely related to) the Lagrangian displacement for the disturbance, e is the disturbance energy, and d represents a variation of the type normally associated with Hamilton's principle. (In fact, d represents a perturbation of the particle trajectories created by advecting the streamlines in a 'frozen-in' manner, d^1 and d^2 being the first- and second-order variations respectively.) Moreover, as we shall see, $\mathrm{d}^2 L$ is a function only of $\boldsymbol{\eta}$ and so we have

$$\frac{1}{2} \int \dot{\boldsymbol{\eta}}^2 \,\mathrm{d}V = e_0 + \mathrm{d}^2 L(\boldsymbol{\eta}), \quad \mathrm{d}^1 L = 0, \tag{1.5}$$

where e_0 is the initial disturbance energy. If we take $\frac{1}{2} \int \dot{\boldsymbol{\eta}}^2 \,\mathrm{d}V$ as a measure of the disturbance, then stability is ensured whenever $\mathrm{d}^2 L$ is negative for all possible $\boldsymbol{\eta}$. ($||\dot{\boldsymbol{\eta}}||^2$ is then bounded from above by e_0.) That is to say, a sufficient, though not necessary, condition for stability is that L is a maximum at equilibrium. This is an obvious extension of the stability test for static equilibria, such as magnetostatics or stationary

† For MHD flows the body force is $\boldsymbol{J} \times \boldsymbol{B}/\rho$, $\mu \boldsymbol{J} = \nabla \times \boldsymbol{B}$. Here we use $\boldsymbol{H}$ to represent the scaled magnetic field $\boldsymbol{B}/(\rho\mu)^{1/2}$.

stratified fluids, in which,

$$\frac{1}{2}\int \dot{\boldsymbol{\eta}}^2\,\mathrm{d}V = e_0 - \delta^2 V, \quad \delta^1 V = 0. \tag{1.6}$$

Here stability is ensured if V (the potential energy) is a minimum at equilibrium. Rayleigh's circulation criterion, the Rayleigh–Taylor criterion and Bernstein's theory for magnetostatics (Bernstein *et al.* 1958) are all examples of (1.6). (The difference between δ and d variations is explained in detail later. Here we just note that δ is taken to be any physically realizable variation, while d is a special subset of δ.)

In fact, it turns out that (1.4) and (1.5) are quite general and not restricted to systems governed by (1.1), or (1.3), and (1.2). Rather, they hold for any non-dissipative flow in which V is a function of $\boldsymbol{\eta}$ and $\boldsymbol{x}$ alone. Moreover, as we shall see, a less stringent test for stability is simply that e is positive definite.

Now the utility of (1.5) lies in the fact that, as we shall see, $\mathrm{d}^2L(\boldsymbol{\eta})$ is particularly easy to evaluate, so that in practice our criterion is simple to implement. Moreover, the derivation and implementation of our criterion requires no knowledge of the integral invariants of the system. (Classical energy methods in stability theory, such as the Kelvin–Arnold variational principle or the energy-Casimir method require, as a first step, that all integral invariants of the system be identified.) Yet, as we shall see, our criterion encompasses many of the well-known energy theorems for linear stability, such as Rayleigh's circulation criterion, the Rayleigh–Taylor criterion for stratified fluids, Bernstein *et al.*'s principle for magnetostatics, Frieman & Rotenberg's criterion for MHD, and the Kelvin–Arnold variational principle applied to Euler flows and to MHD. We might refer to (1.5) as a principle of maximum (rather than least) action.

In summary then, we claim that an equilibrium solution of a conservative system is stable whenever e is positive definite and this is achieved when L is a maximum under a frozen-in perturbation of the $\boldsymbol{u}$-lines. It is the unifying nature of this stability criterion which is the primary novelty of this paper.

1.3. *Classical energy methods in stability analysis*

Now there are several systematic approaches to developing sufficient conditions for the stability of ideal (conservative) flows. The Kelvin–Arnold variational principle, and the energy-Casimir method are, perhaps, the best known. (See Morrison 1998 for a nice review of these.) Both methods are, in effect, elaborate procedures for constructing an (energy-like) functional which is: (i) quadratic in the disturbance; and (ii) conserved by the linearized dynamics. Provided the resulting integral invariant is non-zero for all possible disturbance shapes, it can be used like a Lyaponov functional to bound the growth of disturbances. (That is to say, if $||\delta\boldsymbol{u}||$ is some norm for the disturbance, and δ^2F a conserved quadratic function of $\delta\boldsymbol{u}$, then the flow will be unstable if $||\delta\boldsymbol{u}||$ grows despite the conservation of δ^2F, and so for instability we require $||\delta\boldsymbol{u}||^2/\delta^2F \to \infty$. Consequently, if there exist bounds of the form $|\delta^2F| \geqslant \lambda\,||\delta\boldsymbol{u}||^2$ for all $\delta\boldsymbol{u}$, then the flow cannot be unstable. In short, stability is ensured if δ^2F is positive or negative definite.) In our theory e will provide the conserved quadratic functional.

However, as we shall see, there exists a third procedure for creating a conserved, quadratic functional. Like the Kelvin–Arnold and energy-Casimir methods it relies (in some sense) on the conservation of energy. However, unlike these other methods, it is the Lagrangian, L, rather than the energy, E, which plays the central role. (Hence the appearance of L in (1.4).) We shall describe this procedure in more detail later, but we might note in passing that it relies on expanding the Lagrangian up to quadratic terms in $\boldsymbol{\eta}$, using Lagrange's equations to discard the first variation in L, and then constructing a conserved Hamiltonian for the truncated system. In order to differentiate our procedure from the Kelvin–Arnold and energy-Casimir methods we must briefly summarize these other approaches. Later we shall show the precise link between (1.4) and the Kelvin–Arnold variational principle.

In the Kelvin–Arnold method the appropriate functional is the disturbance energy $\Delta E = E - E_0$, where E_0 is the energy of the base flow. Evidently, ΔE is conserved by the perturbed flow. However, in order to ensure that ΔE is quadratic in the disturbance it is necessary to insist that $\delta^1 E = 0$. It turns out that this can be achieved by restricting the choice of disturbances to those which conserve the topological (frozen-in) invariants of the flow. (Such perturbations are termed isovortical perturbations in the case of Euler flows, or generalized isovortical perturbations for other systems.) In such cases $\delta^2 E$ provides a conserved, quadratic measure of the disturbance (as far as the linearized dynamics are concerned) and stability to infinitesimal disturbances is then ensured if $\delta^2 E$ is positive or negative definite. The art of applying the Kelvin–Arnold variational principle lies in spotting how to conserve all of the topological (frozen-in) invariants when calculating ΔE, i.e. knowing how to construct the generalized isovortical perturbations. This is readily achieved for Euler flows (Arnold 1966*a*; Moffatt 1986) where it is necessary only to ensure that $\boldsymbol{\Omega}$ is frozen-in during the disturbance. However, it becomes quite intricate when it comes to MHD (Friedlander & Vishik 1990) where it becomes necessary to ensure that $\boldsymbol{B}$ is frozen-in as well as conserve the cross-helicity of $\boldsymbol{B}$ and $\boldsymbol{u}$.

In the energy-Casimir method, on the other hand, the appropriate functional is $A = E + C$ where C (the Casimir) is an integral invariant for the flow which reflects, as generally as possible, the frozen-in (topological) invariants such as helicity, cross-helicity etc. If C is constructed in a sufficiently general way then it is usually possible to choose the precise form of C such that $\delta^1 A = 0$ at equilibrium (i.e. we choose C so that $\delta^1 C = -\delta^1 E$). Linear stability is then ensured if $\delta^2 A$ is positive or negative definite (Arnold 1966*b*; Holm *et al.* 1985). The Kelvin–Arnold and energy-Casimir methods are, in fact, closely related with C playing the role of a Lagrange multiplier, effectively building in the topological constraints required by the Kelvin–Arnold method (Holm *et al.* 1985; Davidson 1998).

The use of conserved, quadratic functionals (which are non-zero for all possible disturbance shapes) to bound the growth of perturbations is often referred to as establishing *formal stability*.

1.4. *A trivial example to illustrate the different energy-based stability criteria*

It is important to differentiate our procedure from the Kelvin–Arnold and energy-Casimir methods. A trivial example taken from mechanics will suffice to show the difference. Consider a particle of mass m moving in a circular orbit of radius r_0 under the influence of the radial force $\boldsymbol{F} = f(r)\hat{\boldsymbol{e}}_r$. Suppose that f has potential V, $f = -V'(r)$ and let $\Gamma = r^2\dot{\theta}$ be the angular momentum of the particle. (We restrict ourselves to two-dimensional motion and use polar coordinates r and θ.) We now perturb the trajectory, $r = r_0 + \eta$, $\theta = \theta_0 + \zeta/r_0$, and examine the linear stability of the perturbed trajectory. For this simple system a conventional perturbation analysis provides the necessary and sufficient conditions for stability. The flow is stable if and only if $\lfloor r_0^3 V_0'' + 3r_0^2 V_0' \rfloor > 0$.

Let us now see if we can obtain the same information using the energy principles described above. The energy of the particle on the perturbed path is

$$E = T + V = \tfrac{1}{2}m(\dot{r}^2 + r^2\dot{\theta}^2) + V = E_0 + \delta^1 E + \delta^2 E + \cdots,$$

where δ^1 and δ^2 represents terms which are linear and quadratic in the disturbance respectively. For arbitrary values of η and ζ, $\delta^1 E$ is non-zero. Thus, despite the conservation of E, $\delta^2 E$ does not, in general, provide a conserved, quadratic measure of the disturbance. (Remember, formal stability requires that we can find a conserved, quadratic measure of the disturbance which is positive or negative definite.) In the Kelvin–Arnold procedure we remedy this as follows. We note that the particle conserves not only E but also Γ. We now restrict ourselves to initial perturbations in which $\delta\Gamma = 0$. Since $\delta\Gamma = 0$ at $t = 0$, it must remain zero for all t. Thus we write E as

$$E = \tfrac{1}{2}m(\dot{r}^2 + \Gamma^2/r^2) + V(r)$$

and treat Γ as a constant, $\Gamma = \Gamma_0$. For this restricted set of disturbances we find $\delta^1 E = 0$ and $\delta^2 E = \tfrac{1}{2}m\dot{\eta}^2 + \tfrac{1}{2}\eta^2[V'' + 3V'/r]_0$. In this case conservation of E does indeed ensure that $\delta^2 E$ is conserved by the disturbance (to quadratic order) and so we have formal stability if $\delta^2 E > 0$ for all possible η and $\dot{\eta}$. Thus stability is ensured if $\lfloor r^3 V'' + 3r^2 V' \rfloor_0 > 0$, which coincides with our conventional perturbation analysis. Note that the Kelvin–Arnold method only provides a stability criterion for a restricted set of perturbations (in this case ones where $\delta\Gamma = 0$), although it is readily verified that the value of $\delta\Gamma$ at $t = 0$ does not influence the stability of the perturbed trajectory.

The energy-Casimir method also requires that we spot that Γ is conserved by the particle, although this time there is no need to restrict the form of the initial disturbance. It proceeds as follows. We introduce the generalized invariant, $A = E + C(\Gamma)$ where C is an arbitrary function of Γ (a Casimir). We now choose C such that $\delta^1 A = 0$ for all possible choices of η and ζ. (This requires $C = -m\theta_0\Gamma$.) It follows that $\delta^2 A$ is conserved by the motion. It is readily confirmed that

$$\delta^2 A = \tfrac{1}{2}m(\dot{\eta}^2 + \dot{\zeta}^2) + \tfrac{1}{2}\eta^2[V_0'' - V_0'/r_0].$$

We have formal stability if $\delta^2 A > 0$ for all (η, ζ) and this requires that $V_0'' - V_0'/r_0 > 0$. This coincides with our perturbation analysis since $V_0'' - V_0'/r_0 > 0$ ensures that $V_0'' + 3V_0'/r_0 > 0$. Thus the energy-Casimir method has provided a sufficient (though not necessary) condition for stability.

Our (third) approach does not require that the Casimir invariants of the system (in this case Γ) be identified, although it still relies on the conservation of energy. We proceed as follows. Let $L = T - V$ and η and ζ be generalized coordinates, q_i. We now evaluate,

$$L = L_0 + \delta^1 L + \delta^2 L + \cdots$$

and calculate the generalized momenta, $p_i = \partial L/\partial \dot{q}_i$. The final step is to evaluate the Hamiltonian, H,

$$H = \sum (p_i \dot{q}_i) - L.$$

Since L is not an explicit function of time, H is an invariant. It turns out that

$$e = H + L_0 = \tfrac{1}{2}m(\dot{\eta}^2 + \dot{\zeta}^2) + \tfrac{1}{2}\eta^2[V_0'' - V_0'/r_0].$$

Once again we have a conserved quadratic measure of the disturbance and the motion is stable provided that e is positive definite.

Now in this simple example our third procedure offers no obvious advantage over the others. However, when it comes to more complex systems, where it is by no means obvious what the Casimir invariants are, it does provide an advantage, as we shall see.

2. A general linear stability criterion for conservative systems

We shall now show how this third procedure generalizes to fluid equilibria. The key is to identify an appropriate set of generalized coordinates for the disturbance. At first sight it might seem that the Lagrangian particle displacement (that is, the displacement of the particles from their equilibrium trajectories) should provide a suitable set of generalized coordinates. However, this is not the case. The problem is a subtle one and relates to the fact that the Lagrangian particle displacement field, $\boldsymbol{\zeta}(\boldsymbol{x})$, is not solenoidal. We can remedy this by introducing a closely related field, usually called the *virtual displacement field*, which is defined as $\boldsymbol{\eta} = \boldsymbol{\zeta} - \frac{1}{2}\boldsymbol{\zeta} \cdot \nabla\boldsymbol{\zeta} + \mathrm{HOT}$. We shall see that this second field has a simple physical interpretation and is indeed solenoidal. In order to illustrate the differences between $\boldsymbol{\eta}$ and $\boldsymbol{\zeta}$, and to see how they are used to calculate perturbations in energy, it is useful to consider first the trivial problem of the stability of a static magnetic field. We then generalize the method to any conservative system and to equilibria which are steady rather than simply static.

2.1. *A simple example: the stability of a static magnetic field*

Consider the magnetostatic equilibrium of an ideal fluid. The fluid and magnetic field, $\boldsymbol{B}$, are both assumed to be contained in a volume, V, with a solid surface, S, and the equilibrium is governed by

$$\boldsymbol{J}_0 \times \boldsymbol{B}_0 = \nabla P_0, \quad \boldsymbol{B}_0 \cdot \mathrm{d}\boldsymbol{S} = 0. \tag{2.1}$$

Here the subscript 0 indicates a steady, base configuration whose stability is in question, and $\mathrm{d}\boldsymbol{S}$ is an element of the boundary, S. Now suppose that this equilibrium is slightly disturbed, and that during the initial disturbance the magnetic field is frozen into the fluid. Let $\zeta(\boldsymbol{x}, t)$ be the displacement of a particle, p, from its equilibrium position $\boldsymbol{x}$,

$$\boldsymbol{\zeta}(\boldsymbol{x}, t) = \boldsymbol{x}_p(t) - \boldsymbol{x}_p(0), \quad \boldsymbol{x}_p(0) = \boldsymbol{x}.$$

For $t > 0$, $\boldsymbol{B}$ will be frozen into the fluid and the resulting velocity field, $\boldsymbol{u}(\boldsymbol{x}, t)$, is related to the instantaneous particle displacement, $\boldsymbol{\zeta}$, by

$$\frac{\partial \boldsymbol{\zeta}}{\partial t} = \boldsymbol{u}(\boldsymbol{x} + \boldsymbol{\zeta}, t) = \boldsymbol{u}(\boldsymbol{x}) + \boldsymbol{\zeta} \cdot \nabla \boldsymbol{u} + \cdots . \tag{2.2}$$

Let us now evaluate the change in magnetic energy, E_B, which results from the particle displacement, $\boldsymbol{\zeta}$. We first expand E_B in a series

$$E_B(\boldsymbol{\zeta}, t) = \int (\boldsymbol{B}^2/2\mu)\,\mathrm{d}V = E_{B0} + \delta^1 E_B + \delta^2 E_B + \cdots .$$

Here $\delta^1 E_B$ and $\delta^2 E_B$ are the first- and second-order changes in E_B, $\boldsymbol{\zeta}$ being assumed small at all times. We shall shortly see that $\delta^1 E_B = 0$, while the stability of the magnetostatic equilibrium is determined by the sign of $\delta^2 E_B$. The question, then, is how to evaluate $\delta^1 E_B$ and $\delta^2 E_B$. We now employ a trick. E_B depends only on the instantaneous position of the fluid particles and not their previous histories. That is, E_B is completely determined by the instantaneous spatial distribution of $\boldsymbol{B}$. There are many ways in which each particle could get from $\boldsymbol{x}$ to $\boldsymbol{x}+\boldsymbol{\zeta}$, but, since E_B does not care about the history of the particles, we shall consider the simplest. Following Moffatt (1986), we suppose that for a short time τ we apply an imaginary, steady, solenoidal velocity field, $\boldsymbol{v}(\boldsymbol{x})$, to the fluid. We choose $\boldsymbol{v}(\boldsymbol{x})$ such that it shifts the fluid from its equilibrium configuration to $\boldsymbol{x} + \boldsymbol{\zeta}$. Since the fluid is incompressible $\boldsymbol{v}(\boldsymbol{x})$ must be solenoidal. Now $\boldsymbol{B}$ is frozen into the fluid during the application of $\boldsymbol{v}$ and so we have

$$\frac{\partial \boldsymbol{B}}{\partial t} = \nabla \times (\boldsymbol{v} \times \boldsymbol{B}), \quad 0 < t < \tau. \tag{2.3}$$

It follows that the first- and second-order changes in $\boldsymbol{B}$ are

$$\delta^1 \boldsymbol{B} = \nabla \times (\boldsymbol{\eta} \times \boldsymbol{B}_0), \quad \delta^2 \boldsymbol{B} = \tfrac{1}{2}\nabla \times (\boldsymbol{\eta} \times \delta^1 \boldsymbol{B}), \tag{2.4a,b}$$

where $\boldsymbol{\eta} = \boldsymbol{v}\tau$. This new fields satisfies $\nabla \cdot \boldsymbol{\eta} = 0$ and $\boldsymbol{\eta} \cdot \mathrm{d}\boldsymbol{S} = 0$. It is called the *virtual displacement field* (Moffatt 1986). However, $\boldsymbol{\eta}$ and $\boldsymbol{\zeta}$ are not identical: from (2.2)

$$\boldsymbol{\zeta} = \boldsymbol{\eta} + \tfrac{1}{2}\boldsymbol{\eta} \cdot \nabla \boldsymbol{\eta} + \cdots, \quad \boldsymbol{\eta} = \boldsymbol{\zeta} - \tfrac{1}{2}\boldsymbol{\zeta} \cdot \nabla \boldsymbol{\zeta} + \cdots . \tag{2.5a,b}$$

Thus, the particle displacement and the virtual displacement are equal only at first order. (This turns out to be of crucial importance in our derivation of (1.4).) Let us now evaluate the changes in E_B which results from the application of $\boldsymbol{\eta}$. The first-order change is

$$\delta^1 E_B = \frac{1}{\mu}\int (\boldsymbol{B}_0 \cdot \delta^1 \boldsymbol{B})\,\mathrm{d}V = \frac{1}{\mu}\int \boldsymbol{B}_0 \cdot \nabla \times [\boldsymbol{\eta} \times \boldsymbol{B}_0]\,\mathrm{d}V. \tag{2.6}$$

However the integrand may be rewritten in terms of divergences which, in view of (2.1), integrate to zero. The second-order change in E_B is

$$\delta^2 E_B = \frac{1}{\mu}\int \left[\tfrac{1}{2}(\delta^1 \boldsymbol{B})^2 + \boldsymbol{B}_0 \cdot \delta^2 \boldsymbol{B}\right]\mathrm{d}V,$$

from which

$$\delta^2 E_B = \frac{1}{2\mu}\int \left[\boldsymbol{b}^2 + \boldsymbol{B}_0 \cdot \nabla \times [\boldsymbol{\eta} \times \boldsymbol{b}]\right]\,\mathrm{d}V, \quad \boldsymbol{b} = \nabla \times [\boldsymbol{\eta} \times \boldsymbol{B}_0]. \tag{2.7}$$

This expression gives us the instantaneous perturbation in magnetic energy and magnetic field (to leading order) in terms of the virtual displacement field, $\boldsymbol{\eta}(x,t)$. Now the total energy, $E = T + E_B$, is conserved in ideal MHD. It follows that, for our perturbed magnetostatic equilibrium,

$$E - E_0 = \frac{1}{2}\int [\rho \boldsymbol{u}^2]\,\mathrm{d}V + \delta^2 E_B = \text{constant} \tag{2.8}$$

(cubic- and higher-order terms have been neglected here). We also have, to leading order in $\boldsymbol{\eta}$, $\boldsymbol{u}(\boldsymbol{x},t) = \dot{\boldsymbol{\eta}}(\boldsymbol{x},t)$. Conservation of energy therefore gives us

$$\int \left[\tfrac{1}{2}\rho \dot{\boldsymbol{\eta}}^2\right]\mathrm{d}V + \delta^2 E_B(\boldsymbol{\eta}) = \text{constant} = \Delta E, \tag{2.9a}$$

where $\dot{\boldsymbol{\eta}}$ indicates a partial derivative with respect to time. We now take as our definition of stability the condition that the kinetic energy of the disturbance is always bounded from above by the initial energy of the disturbance, ΔE. It follows that an equilibrium is stable if $\delta^2 E_B$ is positive for all possible shapes of disturbances. That is to say, stability is ensured if $\delta^2 E_B > 0$ for all possible $\boldsymbol{\eta}$. Note that we could rewrite (2.9a) as

$$e = \frac{1}{2}\rho \int \dot{\boldsymbol{\eta}}^2\,\mathrm{d}V + \delta^2 V(\boldsymbol{\eta}) = \text{constant}, \quad \delta^1 V(\boldsymbol{\eta}) = 0, \tag{2.9b}$$

which now becomes a special case of (1.4).

2.2. *A general stability criterion based on the Lagrangian*

We now show that, for any conservative incompressible system,

$$e = \frac{1}{2}\int \dot{\boldsymbol{\eta}}^2 \,\mathrm{d}V - \mathrm{d}^2L(\boldsymbol{\eta}) = \text{constant}, \tag{2.10}$$

which is a generalization of (2.9*b*). The proof relies on expanding the Lagrangian up to second order in the particle displacements, invoking Lagrange's equation to dispense with the first variation in L, and then performing a transformation to create a conserved Hamiltonian, which is quadratic in the disturbance. The first and most important step is to introduce the Lagrangian displacement,

$$\boldsymbol{\zeta}(\boldsymbol{x}, t) = \boldsymbol{x}_p(t) - \boldsymbol{x}_{p0}(t)$$

where $\boldsymbol{x}_{p0}$ is the position vector of particle p in the base flow and $\boldsymbol{x}_p$ is the position of the same particle in the perturbed flow. The generalization of (2.2) is then

$$\frac{\partial \boldsymbol{\zeta}}{\partial t} + \boldsymbol{u}_0(\boldsymbol{x}) \cdot \nabla \boldsymbol{\zeta} = \frac{\mathrm{D}\boldsymbol{\zeta}}{\mathrm{D}t} = \boldsymbol{u}(\boldsymbol{x} + \boldsymbol{\zeta}, t) - \boldsymbol{u}_0(\boldsymbol{x}). \tag{2.11}$$

In the linear (small-amplitude) approximation, this becomes

$$\frac{\partial \boldsymbol{\zeta}}{\partial t} + \boldsymbol{u}_0 \cdot \nabla \boldsymbol{\zeta} = \delta \boldsymbol{u}(\boldsymbol{x}, t) + \boldsymbol{u}_0(\boldsymbol{x} + \boldsymbol{\zeta}) - \boldsymbol{u}_0(\boldsymbol{x}), \tag{2.12}$$

which, using the approximation $\boldsymbol{u}_0(\boldsymbol{x} + \boldsymbol{\zeta}) - \boldsymbol{u}_0(\boldsymbol{x}) = \boldsymbol{\zeta} \cdot \nabla \boldsymbol{u}_0$, simplifies to

$$\delta^1 \boldsymbol{u} = \dot{\boldsymbol{\zeta}} + \nabla \times [\boldsymbol{\zeta} \times \boldsymbol{u}_0]. \tag{2.13}$$

The key step is now to switch from $\boldsymbol{\zeta}$ to $\boldsymbol{\eta}(\boldsymbol{x}, t)$. This greatly simplifies the subsequent analysis. Since $\boldsymbol{\eta}$ and $\boldsymbol{\zeta}$ are equal to leading order, (2.13) yields

$$\delta^1 \boldsymbol{u} = \dot{\boldsymbol{\eta}}(\boldsymbol{x}, t) + \nabla \times [\boldsymbol{\eta} \times \boldsymbol{u}_0]. \tag{2.14}$$

Returning to (2.11), but retaining terms up to second order, we find

$$\delta^2 \boldsymbol{u} = \tfrac{1}{2}\nabla \times [\boldsymbol{\eta} \times \dot{\boldsymbol{\eta}}] + \tfrac{1}{2}\nabla \times [\boldsymbol{\eta} \times (\nabla \times (\boldsymbol{\eta} \times \boldsymbol{u}_0))]. \tag{2.15}$$

We now introduce some notation. We take δ to represent an arbitrary (physically realizable) variation of some field, say $\delta \boldsymbol{u}$. We take d, on the other hand, to represent a frozen-in variation of any field. In the case of the $\boldsymbol{H}$-field, the two coincide ($\delta \boldsymbol{H} = \mathrm{d}\boldsymbol{H}$) since (1.2) demands that, if $\boldsymbol{H}$ is frozen into the fluid during the initial perturbation, then it is frozen-in for all subsequent time. In the case of $\boldsymbol{u}$, however, $\mathrm{d}\boldsymbol{u}$ does not represent a dynamically meaningful perturbation. Nevertheless, we are still free to ask what happens to $\boldsymbol{u}$ and T in the event of a variation in which the $\boldsymbol{u}$-lines are frozen-in. What we choose to do with that information is another matter. From (2.4), (2.6) and (2.7) we have, in terms of the virtual displacement field,

$$\mathrm{d}^1 \boldsymbol{u} = \nabla \times (\boldsymbol{\eta} \times \boldsymbol{u}_0), \quad \mathrm{d}^2 \boldsymbol{u} = \tfrac{1}{2}\nabla \times [\boldsymbol{\eta} \times \mathrm{d}^1 \boldsymbol{u}], \tag{2.16a,b}$$

$$\mathrm{d}^1 T = \int (\boldsymbol{u}_0 \cdot \mathrm{d}^1 \boldsymbol{u}) \,\mathrm{d}V, \quad \mathrm{d}^2 T = \frac{1}{2}\int \left[(\mathrm{d}^1 \boldsymbol{u})^2 + 2\boldsymbol{u}_0 \cdot \mathrm{d}^2 \boldsymbol{u}\right] \mathrm{d}V. \tag{2.17a,b}$$

Evidently,

$$\delta^1 \boldsymbol{u} = \dot{\boldsymbol{\eta}} + \mathrm{d}^1 \boldsymbol{u}, \quad \delta^2 \boldsymbol{u} = \tfrac{1}{2}\nabla \times [\boldsymbol{\eta} \times \dot{\boldsymbol{\eta}}] + \mathrm{d}^2 \boldsymbol{u}. \tag{2.18}$$

(The equivalent expressions in terms of $\boldsymbol{\zeta}$ are far more complicated.) We shall return to these expressions shortly. In the meantime, let us try to understand the significance of d-perturbation as applied to $\boldsymbol{u}$. We shall use the term 'd-*variation*' to mean a perturbation of the equilibrium configuration in which: (i) $\boldsymbol{u}$ is perturbed according to (2.16*a, b*), i.e. the $\boldsymbol{u}$-lines are frozen-in during the perturbations; and (ii) any auxiliary field, such as $\boldsymbol{H}$ and Φ in (1.1), are perturbed in a manner compatible with the auxiliary equations (1.2), i.e. $\boldsymbol{H}$ is frozen-in and Φ is materially conserved. (This requires that the perturbations in $\boldsymbol{H}$ are given by (2.4) while those of Φ satisfy $\delta^1\Phi = -\boldsymbol{\eta}\cdot\nabla\Phi$.) Also, let us write a generalized version of (1.1)

$$\frac{\partial \boldsymbol{u}}{\partial t} = \boldsymbol{u}\times\boldsymbol{\Omega} - \nabla C + \boldsymbol{f}, \tag{2.19}$$

where $\boldsymbol{f}$ is any conservative body force, such as $-[\boldsymbol{H}\times\boldsymbol{J}]+f\nabla\Phi$. Let V be the potential energy associated with $\boldsymbol{f}$. This could, for example, be magnetic energy, gravitational energy, electrostatic energy or some combination of these. From (2.16*a*) and (2.17),

$$\mathrm{d}^1 T = \int [\boldsymbol{\Omega}_0\cdot(\boldsymbol{\eta}\times\boldsymbol{u}_0)]\,\mathrm{d}V = -\int \boldsymbol{\eta}\cdot\boldsymbol{f}_0\,\mathrm{d}V = \mathrm{d}^1 V. \tag{2.20}$$

It follows that $\mathrm{d}^1 L = 0$ under this type of variation, which is the first hint that there is, in fact, some significance to our 'd-variation'. Actually, this is discussed in Davidson (1998) in the context of two-dimensional flows. The physical significance of $\mathrm{d}\boldsymbol{u}$ is that, by advecting the $\boldsymbol{u}$-lines, we create a new set of particle trajectories with the special property that the time of flight between two fixed points is preserved. This is precisely the sort of perturbation demanded by Hamilton's principle and (in two dimensions) $\mathrm{d}^1 L = 0$ is, in fact, a direct consequence of Hamilton's principle (Davidson 1998). In three dimensions we must do a little more work to explain the significance of $\mathrm{d}^1 L = 0$. Once again it rests on the fact that the time of flight of a fluid particle is preserved by the d-variation. To see that this is so consider the time of flight equation

$$t_B - t_A = \int_A^B \frac{\mathrm{d}\boldsymbol{l}}{|\boldsymbol{u}|} = \frac{1}{\Phi}\int_A^B \mathrm{d}V. \tag{2.21}$$

Here Φ is the volume flux down a stream-tube which surrounds a path-line linking A and B, and $\int \mathrm{d}V$ is the volume of the stream-tube (of rectangular cross-section) that may be constructed from pairs of intersecting stream-surfaces which, in turn, might be locally represented by Clebsch variables. Such stream-surfaces are frozen into the fluid during a d-perturbation and so, as in two dimensions, the time of flight of fluid particles is preserved. This ensures that the first variation of the action integral is zero and it is this which lies behind (2.20).

So the idea of a 'd-variation' has some physical basis. We now examine second variations and this will lead to our stability criterion (2.10). The first step is to calculate $\Delta T = T - T_0$ and $\Delta L = L - L_0$ for an arbitrary (physically realizable) δ-variation of the equilibrium state. We have

$$\delta^1 T = \int \boldsymbol{u}_0\cdot\delta^1\boldsymbol{u}\,\mathrm{d}V, \quad \delta^2 T = \frac{1}{2}\int \left[(\delta^1\boldsymbol{u})^2 + 2\boldsymbol{u}_0\cdot\delta^2\boldsymbol{u}\right]\mathrm{d}V.$$

Next, using (2.18) to substitute for $\delta^1\boldsymbol{u}$ and $\delta^2\boldsymbol{u}$, we find

$$\delta^1 T = \mathrm{d}^1 T(\boldsymbol{\eta}) + \int \boldsymbol{u}_0\cdot\dot{\boldsymbol{\eta}}\,\mathrm{d}V, \tag{2.22}$$

$$\delta^2 T = \mathrm{d}^2 T(\boldsymbol{\eta}) + \frac{1}{2}\int \dot{\boldsymbol{\eta}}^2\,\mathrm{d}V + \hat{I}(\boldsymbol{\eta},\dot{\boldsymbol{\eta}}), \tag{2.23}$$

where $\hat{I}$ is bi-linear in $\boldsymbol{\eta}$ and $\dot{\boldsymbol{\eta}}$ and is given by

$$\hat{I} = \frac{1}{2}\int \dot{\boldsymbol{\eta}} \cdot [2\mathrm{d}^1\boldsymbol{u} + \boldsymbol{\Omega}_0 \times \boldsymbol{\eta}]\,\mathrm{d}V. \tag{2.24}$$

Now if $\boldsymbol{f}$ is conservative then the potential energy, V, will depend only on the instantaneous configuration of the flow and not on its history. Thus,

$$\Delta V = V - V_0 = \delta^1 V(\boldsymbol{\eta}) + \delta^2 V(\boldsymbol{\eta}) + \mathrm{HOT}.$$

This gives us an expression for ΔL in terms of $\boldsymbol{\eta}$ and $\dot{\boldsymbol{\eta}}$:

$$\Delta L = \frac{1}{2}\int \dot{\boldsymbol{\eta}}^2\,\mathrm{d}V + \left[\mathrm{d}^1 T(\boldsymbol{\eta}) - \delta^1 V(\boldsymbol{\eta})\right] + \left[\mathrm{d}^2 T(\boldsymbol{\eta}) - \delta^2 V(\boldsymbol{\eta})\right] + I(\boldsymbol{\eta},\dot{\boldsymbol{\eta}}) + \mathrm{HOT},$$

where $I(\boldsymbol{\eta},\dot{\boldsymbol{\eta}}) = \hat{I} + \int \boldsymbol{u}_0 \cdot \dot{\boldsymbol{\eta}}\,\mathrm{d}V$. Now recall that we defined our 'd-variation' such that $\boldsymbol{u}$ is perturbed according to (2.16a,b), but the auxiliary fields, such as $\boldsymbol{H}$ and Φ, are varied in any physically realizable manner. (This requires that $\boldsymbol{H}$ is frozen-in and Φ is materially conserved.) It follows that, as a matter of notation, we can write $\delta^1 V = \mathrm{d}^1 V$ and $\delta^2 V = \mathrm{d}^2 V$. Our expression for the Lagrangian becomes

$$\Delta L = \frac{1}{2}\int \dot{\boldsymbol{\eta}}^2\,\mathrm{d}V + \left[\mathrm{d}^1 T(\boldsymbol{\eta}) - \mathrm{d}^1 V(\boldsymbol{\eta})\right] + \left[\mathrm{d}^2 T(\boldsymbol{\eta}) - \mathrm{d}^2 V(\boldsymbol{\eta})\right] + I(\boldsymbol{\eta},\dot{\boldsymbol{\eta}}).$$

We now use $\boldsymbol{\eta}$ as a set of generalized coordinates describing the instantaneous state of the system. Note that ΔL is a function only of $\boldsymbol{\eta}$ and $\dot{\boldsymbol{\eta}}$. It is not an explicit function of time. Now for a system with a finite number of degrees of freedom, η_i, we have

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial L}{\partial \dot{\eta}_i}\right) - \frac{\partial L}{\partial \eta_i} = 0, \tag{2.25}$$

so that steady solutions are represented by $\partial L/\partial \eta_i = 0$. Also if $L = L(\eta_i, \dot{\eta}_i)$ is not an explicit function of time the system possesses a conserved Hamiltonian,

$$e = \dot{\eta}_i \frac{\partial L}{\partial \dot{\eta}_i} - L = \text{constant}.$$

The equivalent results for our continuous system are that $\mathrm{d}^1 L = 0$ for an equilibrium solution and

$$e = \left[\int \dot{\boldsymbol{\eta}}^2\,\mathrm{d}V + I(\boldsymbol{\eta},\dot{\boldsymbol{\eta}})\right] - \Delta L = \text{constant}.$$

The fact that $\mathrm{d}^1 L = 0$ follows directly from Lagrange's equations is reassuring since (for two-dimensional flows) we have already noted that this may be deduced from Hamilton's principle. Next, substituting for ΔL yields, at last, our conserved, quadratic functional:

$$e = \frac{1}{2}\int \dot{\boldsymbol{\eta}}^2\,\mathrm{d}V - \mathrm{d}^2 L(\boldsymbol{\eta}) = \text{constant}, \quad \mathrm{d}^1 L(\boldsymbol{\eta}) = 0, \tag{2.26}$$

$$\mathrm{d}^2 L(\boldsymbol{\eta}) = \frac{1}{2}\int \left[(\mathrm{d}^1\boldsymbol{u})^2 + \boldsymbol{u}_0 \cdot \nabla \times (\boldsymbol{\eta} \times \mathrm{d}^1\boldsymbol{u})\right]\mathrm{d}V - \delta^2 V(\boldsymbol{\eta}). \tag{2.27}$$

This is the central result of our paper. Since e is a conserved quadratic measure of the disturbance many stability criteria may be established on the back of (2.26). We might refer to (2.26) as a principle of maximum action.

The following two theorems follow directly from (2.26) and (2.27).

THEOREM 1. *The equilibrium of any conservative, incompressible flow possesses (formal) stability provided that*

$$\mathrm{d}^2L(\boldsymbol{\eta}) = \frac{1}{2}\int \left[(\mathrm{d}^1\boldsymbol{u})^2 + \boldsymbol{u}_0\cdot\nabla\times(\boldsymbol{\eta}\times\mathrm{d}^1\boldsymbol{u})\right]\,\mathrm{d}V - \delta^2V(\boldsymbol{\eta}), \quad \mathrm{d}^1\boldsymbol{u} = \nabla\times(\boldsymbol{\eta}\times\boldsymbol{u}_0)$$

is negative definite for all possible $\boldsymbol{\eta}$.

THEOREM 2. *The equilibrium of any conservative, incompressible flow possesses (formal) stability provided that*

$$e = \frac{1}{2}\int \dot{\boldsymbol{\eta}}^2\,\mathrm{d}V - \mathrm{d}^2L(\boldsymbol{\eta}), \quad \dot{\boldsymbol{\eta}} = \delta^1\boldsymbol{u} - \nabla\times(\boldsymbol{\eta}\times\boldsymbol{u}_0)$$

is positive or negative definite for all possible perturbations of the equilibrium.

We shall see shortly that special cases of Theorem 1 are Rayleigh's circulation criterion, the Rayleigh–Taylor criterion for stratified fluids, Bernstein's principle for magnetostatics, and Friedlander & Vishik and Frieman & Rottenberg's stability test for ideal MHD equilibria. A special case of Theorem 2 is Arnold's variational principle for Euler flows.

3. The governing equation for the virtual displacement field

Note that (2.25) also furnishes the governing equation for $\boldsymbol{\eta}(\boldsymbol{x},t)$. Such an equation is useful as it allows us to show the relationship between our criterion and the classical results of Rayleigh, Kelvin, Arnold and others. Substituting for ΔL in (2.25) yields

$$\ddot{\boldsymbol{\eta}} + 2\boldsymbol{u}_0\cdot\nabla\dot{\boldsymbol{\eta}} = \nabla(\dot{\boldsymbol{\eta}}\cdot\boldsymbol{u}_0) + \boldsymbol{F}(\boldsymbol{\eta}), \tag{3.1}$$

where

$$F_i(\boldsymbol{\eta}) = \frac{\delta[\mathrm{d}^2L(\boldsymbol{\eta})]}{\delta\eta_i}. \tag{3.2}$$

The form of $\boldsymbol{F}(\boldsymbol{\eta})$ depends on the nature of the body force. There are two cases of particular interest here: $\boldsymbol{f} = \boldsymbol{J}\times\boldsymbol{H}$ and $\boldsymbol{f} = f\nabla\Phi$. When $\boldsymbol{f} = \boldsymbol{J}\times\boldsymbol{H}$, as in ideal MHD, the $\boldsymbol{H}$-field is frozen-in during the perturbation and we have

$$\delta^2V(\boldsymbol{\eta}) = \frac{1}{2}\int \left[(\mathrm{d}^1\boldsymbol{H})^2 + \boldsymbol{H}_0\cdot\nabla\times(\boldsymbol{\eta}\times\mathrm{d}^1\boldsymbol{H})\right]\,\mathrm{d}V, \quad \mathrm{d}^1\boldsymbol{H} = \nabla\times(\boldsymbol{\eta}\times\boldsymbol{H}_0). \tag{3.3}$$

(This is just δ^2E_B given by (2.7).) In this case (3.2) yields, after a little algebra,

$$\boldsymbol{F} = \boldsymbol{u}_0\times\lfloor\nabla\times\mathrm{d}^1\boldsymbol{u})\rfloor + \mathrm{d}^1\boldsymbol{u}\times[\nabla\times\boldsymbol{u}_0] - \boldsymbol{H}_0\times\left[\nabla\times(\mathrm{d}^1\boldsymbol{H})\right]$$
$$-\mathrm{d}^1\boldsymbol{H}\times[\nabla\times\boldsymbol{H}_0] + \tfrac{1}{2}\nabla(\boldsymbol{\eta}\cdot\nabla C_0). \tag{3.4}$$

(In deriving (3.4) use is made of the fact that

$$\boldsymbol{u}_0\times[\nabla\times(\boldsymbol{\eta}\times\boldsymbol{\Omega}_0)] + [\nabla\times(\boldsymbol{\eta}\times\boldsymbol{u}_0)]\times\boldsymbol{\Omega}_0 + \boldsymbol{J}_0\times[\nabla\times(\boldsymbol{\eta}\times\boldsymbol{H}_0)]$$
$$+[\nabla\times(\boldsymbol{\eta}\times\boldsymbol{J}_0)]\times\boldsymbol{H}_0 = -\nabla(\boldsymbol{\eta}\cdot\nabla C_0), \tag{3.5}$$

which stems directly from the equilibrium condition $\boldsymbol{u}_0\times\boldsymbol{\Omega}_0 + \boldsymbol{J}_0\times\boldsymbol{H}_0 = \nabla C$.)

When the body force is of the form $f(\boldsymbol{x})\nabla\Phi$, on the other hand, we require that Φ is materially conserved during the perturbation and so

$$\delta^1\Phi = -\boldsymbol{\eta}\cdot\nabla\Phi, \quad \delta^2\Phi = -\tfrac{1}{2}\boldsymbol{\eta}\cdot\nabla(\delta^1\Phi).$$

The potential energy is $V = \int f\Phi \, \mathrm{d}V$, from which

$$\delta^2 V(\boldsymbol{\eta}) = -\frac{1}{2}\int (\boldsymbol{\eta}\cdot\nabla\Phi)(\boldsymbol{\eta}\cdot\nabla f)\,\mathrm{d}V. \tag{3.6}$$

In this case (3.2) yields

$$\boldsymbol{F}(\boldsymbol{\eta}) = \boldsymbol{u}_0 \times (\nabla \times \mathrm{d}^1\boldsymbol{u}) + \mathrm{d}^1\boldsymbol{u} \times (\nabla \times \boldsymbol{u}_0) - \delta^1\Phi\nabla f_0 + \nabla(\cdot), \tag{3.7}$$

where $\nabla(\cdot)$ represents a gradient term which is unimportant.

It is readily confirmed that, for both (3.4) and (3.7), $\boldsymbol{F}(\boldsymbol{\eta})$ is self-adjoint in the sense that

$$\int \boldsymbol{\eta}_2 \cdot \boldsymbol{F}(\boldsymbol{\eta}_1)\,\mathrm{d}V = \int \boldsymbol{\eta}_1 \cdot \boldsymbol{F}(\boldsymbol{\eta}_2)\,\mathrm{d}V.$$

Moreover, if we take the dot product of (3.1) with $\dot{\boldsymbol{\eta}}$ and integrate over the domain, we obtain

$$\frac{1}{2}\int \dot{\boldsymbol{\eta}}^2\,\mathrm{d}V - \frac{1}{2}\int \boldsymbol{F}\cdot\boldsymbol{\eta}\,\mathrm{d}V = \text{constant}. \tag{3.8}$$

It follows from (2.26) that

$$W(\boldsymbol{\eta}) = \frac{1}{2}\int \boldsymbol{F}\cdot\boldsymbol{\eta}\,\mathrm{d}V = \mathrm{d}^2L(\boldsymbol{\eta}), \tag{3.9}$$

which completes the link between the linearized force operator, $\boldsymbol{F}(\boldsymbol{\eta})$, and the Lagrangian, $\mathrm{d}^2L(\boldsymbol{\eta})$, which is the basis of our stability criterion. An expression similar to (3.8) was derived by Frieman & Rottenberg (1960) for ideal MHD. We shall now show that the classical results of Rayleigh, Kelvin, Arnold, Bernstein *et al.* and Frieman & Rottenberg are all special cases of (2.26). We start with Euler flows, in which $\boldsymbol{f} = \boldsymbol{0}$.

4. Euler flows and Arnold's theorem: a special case for our criterion

We now show that the Kelvin–Arnold variational principle, as applied to Euler flows (Arnold 1966*a*), is a special case of (2.26). We start by noting that, when $\boldsymbol{f} = 0$, (3.1) becomes

$$\ddot{\boldsymbol{\eta}} + 2\boldsymbol{u}_0\cdot\nabla\dot{\boldsymbol{\eta}} = \boldsymbol{u}_0 \times (\nabla \times \mathrm{d}^1\boldsymbol{u}) + \mathrm{d}^1\boldsymbol{u} \times (\nabla \times \boldsymbol{u}_0) + \nabla(\cdot). \tag{4.1}$$

This may be integrated once to give

$$\dot{\boldsymbol{\eta}} = \boldsymbol{\eta} \times \boldsymbol{\Omega}_0 - \nabla \times (\boldsymbol{\eta} \times \boldsymbol{u}_0) + \boldsymbol{m} + \nabla(\cdot), \tag{4.2}$$

where $\boldsymbol{m}$ is independent of $\boldsymbol{\eta}$ and is governed by

$$\partial \boldsymbol{m}/\partial t = \nabla \times (\boldsymbol{u}_0 \times \boldsymbol{m}).$$

It follows from (2.18) that

$$\delta^1\boldsymbol{u} = \boldsymbol{\eta} \times \boldsymbol{\Omega}_0 + \boldsymbol{m} + \nabla(\cdot). \tag{4.3}$$

If, at $t = 0$, we specify that $\boldsymbol{m} = 0$, then $\boldsymbol{m}$ will be zero for all time. In such a case

$$\delta^1\boldsymbol{\Omega} = \nabla \times (\boldsymbol{\eta} \times \boldsymbol{\Omega}_0). \tag{4.4}$$

Evidently, this is a perturbation in which the $\boldsymbol{\Omega}$-lines are frozen into the fluid – an *isovortical perturbation*. The Kelvin–Arnold principle states that a steady Euler flow is stable provided that $\delta^2 T$ is positive definite or negative definite under an isovortical perturbation. Let us denote such a perturbation by $\hat{\mathrm{d}}$, to distinguish it from a general perturbation, δ. However, using (2.18) it is readily confirmed that

$$\hat{\mathrm{d}}^2 T = \frac{1}{2}\int \dot{\boldsymbol{\eta}}^2 \,\mathrm{d}V - \mathrm{d}^2 T = e. \tag{4.5}$$

Thus the Kelvin–Arnold principle is simply a special case of Theorem 2. We now turn to flows in which $\boldsymbol{f}$ is non-zero.

5. More special cases of our criterion: classical stability criteria for forced flows

We now explore the consequences of (2.26) for flows subject to a body force. We shall see that Bernstein's theorem, Rayleigh's circulation criterion for swirling flow, the Rayleigh–Taylor criterion for stratified fluid, and Arnold's variational theorem applied to MHD all follow directly from (2.26). We start with static equilibria, in which $\boldsymbol{u}_0 = 0$.

5.1. *Static equilibria*

If $\boldsymbol{u}_0 = 0$ then (2.26) simplifies to

$$e = \frac{1}{2}\int \dot{\boldsymbol{\eta}}^2 \,\mathrm{d}V + \mathrm{d}^2 V(\boldsymbol{\eta}) = \text{constant}. \tag{5.1}$$

The integral on the right is now the kinetic energy of the disturbance, while V may, for example, be magnetic energy or gravitational potential energy. Equation (5.1) also applies to the case of an axisymmetric swirling flow, subject to axisymmetric perturbations, $(0, u_\theta(r), 0)$. V is then the potential energy of the centrifugal force, $(\Gamma^2/r^3)\hat{\boldsymbol{e}}_r$, associated with the angular momentum, $\Gamma = ru_\theta$. In such a case $\mathrm{D}\Gamma/\mathrm{D}t = 0$ and so we have a force $f\nabla\Phi - \nabla(\cdot)$ with $\boldsymbol{u}_0 = \boldsymbol{0}$, $f = (2r^2)^{-1}$ and $\Phi = \Gamma^2$. Equation (3.6) then yields (Davidson 1998)

$$\mathrm{d}^2 V = \int \left[\tfrac{1}{2}(\mathrm{d}^1\Gamma)^2 + \Gamma_0 \mathrm{d}^2\Gamma\right] r^{-2}\,\mathrm{d}V = \int \frac{\eta_r^2}{2r^3}\frac{\mathrm{d}\Gamma_0^2}{\mathrm{d}r}\,\mathrm{d}V.$$

Note that the integrand on the right is simply Rayleigh's descriminant. Necessary and sufficient conditions for the stability of magnetostatic equilibria (Bernstein's theorem), swirling flow (Rayleigh's criterion) and stratified fluid, all follow directly from (5.1). In each case we simply require that $V(\boldsymbol{\eta})$ is a minimum at equilibrium. The sufficiency of $\mathrm{d}^2 V > 0$ follows from conservation of energy,

$$\frac{1}{2}\int \dot{\boldsymbol{\eta}}^2\,\mathrm{d}V \leqslant e_0 \quad \text{for all } t.$$

The necessity of $\mathrm{d}^2 V > 0$ (for all $\boldsymbol{\eta}$) requires a little more work. One method of proof is given by Biskamp (1993) for the particular case of magnetostatics, but this is readily generalized to all static equilibria.

The argument is as follows: suppose that $W(\boldsymbol{\eta})$, which is defined by (3.9) and equals $-\mathrm{d}^2 V(\boldsymbol{\eta})$ for static equilibria, is indefinite in sign or else positive definite. Then for some $\boldsymbol{\eta} = \boldsymbol{\eta}^*$ we have

$$W(\boldsymbol{\eta}^*) = \gamma^2 \int \tfrac{1}{2}(\boldsymbol{\eta}^*)^2\,\mathrm{d}V,$$

where γ is a real constant. Next we note that (3.1) is second order in t and so $\boldsymbol{\eta}$ and $\dot{\boldsymbol{\eta}}$ may be specified separately at $t = 0$. We choose $\boldsymbol{\eta}(0) = \boldsymbol{\eta}^*$ and $\dot{\boldsymbol{\eta}}(0) = \gamma\boldsymbol{\eta}^*$. The disturbance energy is then zero and we have

$$\frac{1}{2}\int \dot{\boldsymbol{\eta}}^2 \,\mathrm{d}V = W(\boldsymbol{\eta}). \tag{5.2}$$

We now return to (3.1) which, on multiplication by $\boldsymbol{\eta}$ and integration over V, yields

$$\frac{1}{2}\ddot{I} + \int \boldsymbol{\eta}\cdot(\boldsymbol{u}_0\cdot\nabla\dot{\boldsymbol{\eta}})\,\mathrm{d}V = \frac{1}{2}\int \dot{\boldsymbol{\eta}}^2\,\mathrm{d}V + W(\boldsymbol{\eta}) = \int \dot{\boldsymbol{\eta}}^2\,\mathrm{d}V. \tag{5.3}$$

Here $I = \frac{1}{2}\int(\boldsymbol{\eta}^2)\,\mathrm{d}V$. Now the Schwartz inequality gives us

$$\dot{I}^2 \leqslant 2I\int \dot{\boldsymbol{\eta}}^2\,\mathrm{d}V$$

and so (5.3) can be rewritten as

$$I\ddot{I} - \dot{I}^2 \geqslant 2I\int \dot{\boldsymbol{\eta}}\cdot(\boldsymbol{u}_0\cdot\nabla\boldsymbol{\eta})\,\mathrm{d}V. \tag{5.4}$$

When $\boldsymbol{u}_0 = 0$, the right-hand side of (5.4) is zero. This ensures exponential growth at a rate $I \geqslant I(0)\exp[2\gamma t]$. (This can be seen from making the substitution $y = \ln(I/I(0))$ and integrating the resulting equation, $\ddot{y} \geqslant 0$.) Thus static energy criteria, such as Bernstein's theorem or Rayleigh's circulation criterion, provide necessary as well as sufficient conditions for stability. Note, however, that when $\boldsymbol{u}_0$ is non-zero, (5.4) allows no such deduction.

5.2. *Non-static equilibria in MHD*

We now consider the relationship between (2.26) and the Kelvin–Arnold variational principle applied to MHD. Consider the perturbations

$$\hat{\mathrm{d}}^1\boldsymbol{H} = \nabla\times[\hat{\boldsymbol{\eta}}\times\boldsymbol{H}_0], \tag{5.5}$$

$$\hat{\mathrm{d}}^2\boldsymbol{H} = \tfrac{1}{2}\nabla\times[\hat{\boldsymbol{\eta}}\times\hat{\mathrm{d}}^1\boldsymbol{H}], \tag{5.6}$$

$$\hat{\mathrm{d}}^1\boldsymbol{\Omega} = \nabla\times[\hat{\boldsymbol{\eta}}\times\boldsymbol{\Omega}_0] + \nabla\times[\hat{\boldsymbol{\eta}}^*\times\boldsymbol{H}_0], \tag{5.7}$$

$$2\hat{\mathrm{d}}^2\boldsymbol{\Omega} = \nabla\times\lfloor\hat{\boldsymbol{\eta}}\times\hat{\mathrm{d}}^1\boldsymbol{\Omega}\rfloor + \nabla\times\lfloor\hat{\boldsymbol{\eta}}^*\times\hat{\mathrm{d}}^1\boldsymbol{H}\rfloor. \tag{5.8}$$

Here we have used $\hat{\mathrm{d}}$ and $\hat{\boldsymbol{\eta}}$ rather than δ and $\boldsymbol{\eta}$ since (5.5)–(5.8) represent a restricted set of perturbations not unlike, but different from, the d-variation. As before, $\hat{\boldsymbol{\eta}}$ represents a virtual displacement field. However, now we have a second solenoidal field, $\hat{\boldsymbol{\eta}}^*$, which is independent of $\hat{\boldsymbol{\eta}}$ and need not satisfy $\hat{\boldsymbol{\eta}}^*\cdot\mathrm{d}\boldsymbol{S} = 0$. These equations were first proposed by Friedlander & Vishik (1990) for MHD equilibria and later proposed by Davidson (1998) for two-dimensional solutions of our model system (1.1)–(1.2). Now (5.5)–(5.8) ensure that the $\boldsymbol{H}$-lines are frozen into the fluid during the perturbation. Moreover, it is readily confirmed that they also preserve the cross-helicity of $\boldsymbol{u}$ and $\boldsymbol{H}$ for all material volumes V_H defined by $\boldsymbol{H}\cdot\boldsymbol{n} = 0$ on S_H. We conclude therefore, that (5.5)–(5.8) constitute a generalized isovortical perturbation. Following Arnold (1966*a*) we would expect $\hat{\mathrm{d}}^1E$ to be zero, and indeed this is readily confirmed by direct substitution. We would also expect $\hat{\mathrm{d}}^2E$ to be conserved by the linearized dynamics, and again this may be confirmed by direct substitution. (The proof for two-dimensional motion is given by Davidson 1998 and for three-dimensional MHD by Vladimirov & Ilin 1998.) It follows that a sufficient condition for (formal) stability is that $\hat{\mathrm{d}}^2E$ is of definite sign for all non-trivial $\hat{\boldsymbol{\eta}}$ and $\hat{\boldsymbol{\eta}}^*$.

The question now arises as to the relationship between the Kelvin–Arnold method for MHD and (2.26). It is not difficult to show that, by virtue of (5.7) and (5.8),

$$\hat{\mathrm{d}}^2T(\hat{\boldsymbol{\eta}}) + \mathrm{d}^2T(\hat{\boldsymbol{\eta}}) = \frac{1}{2}\int \dot{\hat{\boldsymbol{\eta}}}^2\,\mathrm{d}V,$$

where, as always, $\dot{\boldsymbol{\eta}} = \delta^1\boldsymbol{u} - \nabla\times(\boldsymbol{\eta}\times\boldsymbol{u}_0)$ and we have restricted $\boldsymbol{\eta}$ to $\hat{\boldsymbol{\eta}}$. Now we also have $\mathrm{d}^2V = \hat{\mathrm{d}}^2V$ since $\boldsymbol{H}$ is frozen into the fluid. It follows that the second variation in E under a $\hat{\mathrm{d}}$-perturbation is

$$\hat{\mathrm{d}}^2E = \hat{\mathrm{d}}^2T + \hat{\mathrm{d}}^2V = \frac{1}{2}\int \dot{\hat{\boldsymbol{\eta}}}^2\,\mathrm{d}V - \mathrm{d}^2L(\hat{\boldsymbol{\eta}}).$$

On comparing this with (2.26) we see that $e = \hat{\mathrm{d}}^2E$ whenever $\boldsymbol{\eta}$ is restricted to $\hat{\boldsymbol{\eta}}$. Thus the Kelvin–Arnold variational principle applied to MHD is a special case of (2.26) with $\hat{\mathrm{d}}^2E$ acting as a Hamiltonian for the truncated system. Note, however, that (2.26) is a stronger result, since the Kelvin–Arnold principle only applies to perturbations which are initially of the generalized isovortical type. In the language of Hamiltonian mechanics, Arnold's variational principle applies only to *symplective leaves* in the function space of $\boldsymbol{u}$ and $\boldsymbol{H}$. Equation (2.26), on the other hand, applies to all points in function space.

We conclude this section with a discussion of instability criteria. It is well-known that Arnold's criterion provides only a sufficient condition for stability. Indeed, many authors have pointed to examples of stable Euler flows in which $\hat{\mathrm{d}}^2E$ is indefinite in sign. By implication a positive d^2L cannot ensure instability. Yet, for the static equilibria discussed above, our stability condition is both necessary and sufficient. Why should this be so? In this respect it is informative to return to Biskamp's proof of the necessity of $\mathrm{d}^2V > 0$ for the stability of static equilibria, adapted now to non-static equilibria. When $\boldsymbol{u}_0$ is non-zero, and $W(\boldsymbol{\eta})$ is positive for some $\boldsymbol{\eta}$ (so that our stability criterion is violated), the steps leading up to (5.4) remain valid. Hence exponential growth of a disturbance is ensured provided the integral on the right of (5.4) is positive or zero. This is certainly the case if $\dot{\boldsymbol{\eta}}$ and $\boldsymbol{\eta}$ have the same spatial structure. Consequently, if the base flow can support normal modes of fixed spatial structure, and $W(\boldsymbol{\eta}) > 0$ for that mode, instability results. In fact, this is self-evident since choosing initial conditions such that $e = 0$ gives, for that mode,

$$\frac{1}{2}\int \dot{\boldsymbol{\eta}}^2\,\mathrm{d}V = W(\boldsymbol{\eta}) = \gamma^2\int \tfrac{1}{2}\boldsymbol{\eta}^2\mathrm{d}V$$

for some γ. This gives exponential growth, $\boldsymbol{\eta} \sim \exp[\gamma t]$. However, it seems unlikely that such modes exist, as the following trivial example illustrates.

Consider the MHD flow $\boldsymbol{u}_0 = \Omega r\hat{\boldsymbol{e}}_\theta$, $\boldsymbol{H}_0 = a\boldsymbol{u}_0$ (for some constant a), in the domain $0 < r < R$. We now examine the stability of this to two-dimensional disturbances, $\boldsymbol{\eta} = \boldsymbol{\eta}(r,\theta)$. It is readily confirmed by direct substitution that the curl of (3.1) yields

$$\left[\frac{\partial}{\partial t} + \Omega(1+a)\frac{\partial}{\partial\theta}\right]\left[\frac{\partial}{\partial t} + \Omega(1-a)\frac{\partial}{\partial\theta}\right]\nabla^2(r\eta_r) = 0. \tag{5.9}$$

The resulting solutions are stable Alfvén waves travelling along the $\boldsymbol{H}$-lines (either clockwise or anti-clockwise) and riding on the back of the base flow:

$$\eta_r = \eta_r(\theta - \Omega(1\pm a)t).$$

If we apply Arnold's criterion to this, or indeed any two-dimensional MHD flow, we find that $\hat{d}^2E$ is indefinite in sign whenever $|\boldsymbol{u}_0| > |\boldsymbol{H}_0|$. (See for example, Davidson 1998 or else § 6.) Thus the flow is stable, despite the fact that $W(\boldsymbol{\eta})$ can adopt positive values. This simple example illustrates that Arnold's criterion (and ours) fails to provide a necessary condition for stability. Moreover, it provides a hint as to why (5.4) and the associated arguments prove fruitless when $\boldsymbol{u}_0 \neq 0$. The key point is that any disturbance will be advected by the base flow as well as propagate with the group velocity of the underlying wave motion (Alfvén waves, inertial waves, etc). It seems unlikely, therefore, that normal modes of fixed spatial structure can be identified and so the right-hand integral in (5.4) will usually be indefinite in sign.

6. Some new results for MHD flow

6.1. *Non-parallel versus co-linear flows*

We now focus on MHD flow, where our conserved, quadratic functional reduces to that of Friedlander & Vishik (1990). It is well-known that there is a fundamental distinction between base flows for which $\boldsymbol{u}_0$ and $\boldsymbol{H}_0$ are parallel and those in which they are non-parallel. Here we show that when $\boldsymbol{u}_0$ and $\boldsymbol{H}_0$ are co-linear we may identify a wide range of stable, steady flows. Moreover, the stability of such flows may be determined very simply by examining the stability of an equivalent magnetostatic equilibrium, constructed in a particular way from $\boldsymbol{u}_0$ and $\boldsymbol{H}_0$. When $\boldsymbol{u}_0$ and $\boldsymbol{H}_0$ are non-parallel on the other hand, d^2L is always indefinite in sign, suggesting instability. Our starting point is (2.27) which, for MHD flow, simplifies to

$$\begin{aligned}\mathrm{d}^2L = -\frac{1}{2}\int \big[(\mathrm{d}^1\boldsymbol{H})^2 + \boldsymbol{H}_0\cdot\nabla\times(\boldsymbol{\eta}\times\mathrm{d}^1\boldsymbol{H}) \\ -(\mathrm{d}^1\boldsymbol{u})^2 - \boldsymbol{u}_0\cdot\nabla\times(\boldsymbol{\eta}\times\mathrm{d}^1\boldsymbol{u})\big]\,\mathrm{d}V,\end{aligned} \tag{6.1}$$

where

$$\mathrm{d}^1\boldsymbol{u} = \nabla\times(\boldsymbol{\eta}\times\boldsymbol{u}_0), \quad \mathrm{d}^1\boldsymbol{H} = \nabla\times(\boldsymbol{\eta}\times\boldsymbol{H}_0).$$

We now note that, when $\boldsymbol{u}_0$ and $\boldsymbol{H}_0$ are non-parallel, d^2L is indeed indefinite in sign. Consider a short-wavelength disturbance for which $kl \gg 1$, k being a typical wavenumber and l a typical scale for $\boldsymbol{u}_0$. Then (6.1) reduces to

$$\mathrm{d}^2L \approx -\frac{1}{2}\int \big[(\boldsymbol{H}_0\cdot\nabla\boldsymbol{\eta})^2 - (\boldsymbol{u}_0\cdot\nabla\boldsymbol{\eta})^2\big]\,\mathrm{d}V. \tag{6.2}$$

Now suppose $\boldsymbol{\eta}$ varies rapidly in the direction of $\boldsymbol{u}_0$ but slowly in the direction of $\boldsymbol{H}_0$; then d^2L is positive. The converse is also true: d^2L is negative when $\boldsymbol{\eta}$ varies slowly in the direction of $\boldsymbol{u}_0$ but rapidly in the direction of $\boldsymbol{H}_0$. Thus, provided $\boldsymbol{u}_0$ and $\boldsymbol{H}_0$ are not parallel, d^2L is always indefinite in sign and our criterion is violated. This was observed by Friedlander & Vishik (1990, 1995) and is the first hint that all such flows are unstable. We return to this idea in § 6.3. First, however, we consider flows in which $\boldsymbol{u}_0$ and $\boldsymbol{H}_0$ are co-linear.

6.2. *Flows in which $\boldsymbol{u}_0$ and $\boldsymbol{H}_0$ are parallel*

We are looking for stable solutions of

$$\boldsymbol{u}_0\times\boldsymbol{H}_0 = 0, \quad \boldsymbol{u}_0\times\boldsymbol{\Omega}_0 - \boldsymbol{H}_0\times\boldsymbol{J}_0 = \nabla C. \tag{6.3a, b}$$

We shall show that stability of system (6.3) can be determined by examining the stability of an equivalent magnetostatic equilibrium, and that for every stable magnetostatic equilibrium we can construct a family of stable, non-static equilibria. From (6.3a) we have

$$\boldsymbol{u}_0 = \alpha \boldsymbol{H}_0, \quad \boldsymbol{H}_0 \cdot \nabla\alpha = 0.$$

Suppose we take α as constant. (We relax this condition later.) Then

$$\boldsymbol{H}_0 \times \boldsymbol{J}_0 = -\nabla\lfloor C/(1-\alpha^2)\rfloor \tag{6.4}$$

and

$$\mathrm{d}^2 L = -\tfrac{1}{2}(1-\alpha^2)\int \left[(\mathrm{d}^1\boldsymbol{H})^2 + \boldsymbol{H}_0 \cdot \nabla \times (\boldsymbol{\eta} \times \mathrm{d}^1\boldsymbol{H})\right] \mathrm{d}V. \tag{6.5}$$

Thus we have constructed a magnetostatic equilibrium, $\boldsymbol{H}_s$, from $(\boldsymbol{u}_0, \boldsymbol{H}_0)$, in which $\boldsymbol{H}_s = \boldsymbol{H}_0$ and $C_s = C/(1-\alpha^2)$. Moreover, $\boldsymbol{H}_s$ is stable if and only if

$$\mathrm{d}^2 L_s = -\frac{1}{2}\int \left[(\mathrm{d}^1\boldsymbol{H})^2 + \boldsymbol{H}_s \cdot \nabla \times (\boldsymbol{\eta} \times \mathrm{d}^1\boldsymbol{H})\right] \mathrm{d}V$$

is negative definite. It follows from (6.4) that $(\boldsymbol{u}_0, \boldsymbol{H}_0)$ is stable provided: (i) $|\boldsymbol{u}_0| < |\boldsymbol{H}_0|$; and (ii) the static magnetic field, $\boldsymbol{H}_s$, is stable. However, there are several stable three-dimensional static equilibria, such as force-free fields. (See for example Moffatt 1986.) By implication there also exist stable three-dimensional non-static equilibria. Specifically, for each stable magnetostatic field, $\boldsymbol{H}_s$, we can construct a stable non-static equilibrium, $\boldsymbol{u}_0 = \alpha\boldsymbol{H}_s$, $\boldsymbol{H}_0 = \boldsymbol{H}_s$.

We now turn to two-dimensional flows. Here the restriction $\alpha =$ constant can be lifted. This is true of planar and poloidal flows although, in the interests of brevity, we consider only the planar case. We shall show that, once again, the stability of an equilibrium may be determined by the stability of an equivalent magnetostatic equilibrium, and that for every stable magnetostatic equilibrium we may construct a family of stable non-static equilibria. We start by introducing Φ and Ψ, the flux-function and stream-function for $\boldsymbol{H}$ and $\boldsymbol{u}$, defined via $\boldsymbol{H} = \nabla \times [\Phi\hat{\boldsymbol{e}}_z]$ and $\boldsymbol{u} = \nabla \times [\Psi\hat{\boldsymbol{e}}_z]$. Then (6.3$a,b$) reduce to

$$\Phi_0 = \Phi_0(\Psi_0), \quad \nabla^2\Phi_0 + C'(\Phi_0) = (\nabla^2\Psi_0)\Psi_0'(\Phi_0), \tag{6.6a,b}$$

while (6.1) yields the well-known functional

$$\mathrm{d}^2 L = -\frac{1}{2}\int \left[(1-(\Psi_0')^2)(\nabla\phi)^2 + g\phi^2\right] \mathrm{d}V. \tag{6.7}$$

Here $\phi = \mathrm{d}^1\Phi = -\boldsymbol{\eta} \cdot \nabla\Phi_0$ and g is (see Davidson 1998)

$$g = \Psi_0'\nabla^2\Psi_0' + \Psi_0''\nabla^2\Psi_0 - C_0''(\Phi_0). \tag{6.8}$$

As noted by several authors, and confirmed by (6.7), we can ensure stability only if $|\Psi_0'| < 1$ and we now restrict ourselves to such cases. Next, consider the flux-function Φ_s, and associated magnetic field, $\boldsymbol{H}_s$, defined by

$$\Phi_s = \int_0^{\Phi} (1-(\Psi_0')^2)^{1/2}\, \mathrm{d}\Phi, \tag{6.9a}$$

$$\boldsymbol{H}_s = (1-(\Psi_0')^2)^{1/2}\boldsymbol{H}_0. \tag{6.9b}$$

Since $|\boldsymbol{u}_0| < |\boldsymbol{H}_0|$, the integrand in (6.9$a$) is real. Substituting (6.9a) into (6.6b) we obtain

$$\nabla^2\Phi_s + C'(\Phi_s) = 0. \tag{6.10}$$

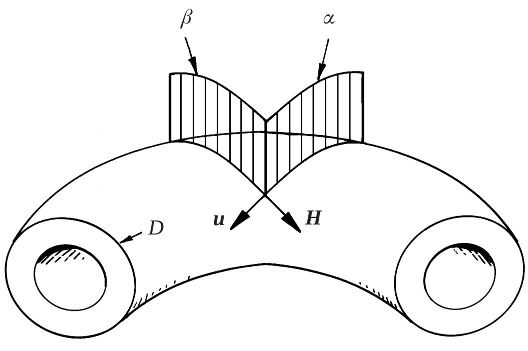

FIGURE 1. Clebsch variables.

On comparison with (6.6*b*) we see that Φ_s represents a magnetostatic equilibrium. Moreover, if we evaluate

$$d^2L_s = -\frac{1}{2}\int \left[(\nabla\phi_s)^2 + g_s\phi_s^2\right] dV, \tag{6.11}$$

where $\phi_s = -\boldsymbol{\eta}\cdot\nabla\Phi_s$ and $g_s = -C''(\Phi_s)$, it is readily demonstrated that $d^2L_s = d^2L$. Now suppose that $\boldsymbol{H}_s$ is stable. Then the stability of the magnetostatic equilibrium ensures d^2L_s is negative definite which, from (6.11), ensures that $(\boldsymbol{u}_0, \boldsymbol{H}_0)$ is stable. Once again, the stability of a non-static equilibrium may be assessed from the stability of the appropriate magnetostatic field.

6.3. *Flows in which $\boldsymbol{u}_0$ and $\boldsymbol{H}_0$ are non-parallel*

We have seen that flows in which $\boldsymbol{u}_0$ and $\boldsymbol{H}_0$ are not parallel violate our stability criterion and so are potentially unstable. We now try to identify some of the unstable modes. We shall consider short-wavelength localized disturbances to $(\boldsymbol{u}_0, \boldsymbol{H}_0)$. We start with the equilibrium equation

$$\boldsymbol{u}_0 \times \boldsymbol{H}_0 = \nabla D. \tag{6.12}$$

Now D is a well-defined single-valued scalar function (Hameiri 1983). Thus the surfaces $D =$ constant are combined stream-function–magnetic-flux surfaces. Let us now assume that both $\boldsymbol{u}_0$ and $\boldsymbol{H}_0$ allow a local representation in terms of Clebsch variables:

$$\boldsymbol{u}_0 = (\nabla\alpha)\times(\nabla D), \quad \boldsymbol{H}_0 = (\nabla\beta)\times(\nabla D). \tag{6.13}$$

Of course, when the helicity of either field is non-zero such a representation cannot be globally valid. Nevertheless, we are concerned here with a local description of the flow, and not global modes of instability, and so (6.13) provides a useful representation of $\boldsymbol{u}_0$ and $\boldsymbol{H}_0$. α and D are the local stream-function surfaces while β and D are the local magnetic-flux surfaces (see figure 1). From (6.12) we have

$$(\nabla\alpha \times \nabla\beta)\cdot\nabla D = 1$$

and it follows from (6.13) that

$$\boldsymbol{u}_0\cdot\nabla\beta = -1, \quad \boldsymbol{H}_0\cdot\nabla\alpha = 1. \tag{6.14a,b}$$

We shall return to these expressions shortly. Now suppose that, at $t = 0$, $\boldsymbol{\eta}$ is zero everywhere except within a small sub-domain of the flow. As long as $\boldsymbol{\eta}$ remains a local disturbance we may use representation (6.13) for $(\boldsymbol{u}_0, \boldsymbol{H}_0)$. Let us now look for solutions of (3.1) in the form of a wave packet. In particular, consider

$$\boldsymbol{\eta} = \hat{\eta}\gamma(\beta, t)\boldsymbol{H}_0, \tag{6.15}$$

where $\hat{\eta}$ is a slowly varying amplitude and $\gamma(\beta, t)$ represents a fast oscillation across the β-planes. We shall assume that the spatial and temporal derivatives of $\hat{\eta}$ are sufficiently small, by comparison with those of γ, that they may be neglected in the following analysis. Then, using γ' to represent the rate of change of γ with respect to β, we have

$$\nabla \times [\boldsymbol{\eta} \times \boldsymbol{u}_0] = -\hat{\eta}\gamma'\boldsymbol{H}_0, \quad \nabla \times [\boldsymbol{\eta} \times \boldsymbol{H}_0] = 0,$$

and expression (3.4) for $\boldsymbol{F}(\boldsymbol{\eta})$ becomes

$$\boldsymbol{F}(\boldsymbol{\eta}) = [-\gamma''\boldsymbol{H}_0 + 2\gamma'\boldsymbol{u}_0 \cdot \nabla\boldsymbol{H}_0 - \nabla(\gamma'\boldsymbol{u}_0 \cdot \boldsymbol{H}_0)]\hat{\eta}.$$

However, from (6.14*a*)

$$\boldsymbol{u}_0 \cdot \nabla(\gamma'\boldsymbol{H}_0) = -\gamma''\boldsymbol{H}_0 + \gamma'\boldsymbol{u}_0 \cdot \nabla\boldsymbol{H}_0$$

and so $\boldsymbol{F}(\boldsymbol{\eta})$ can be rewritten as

$$\boldsymbol{F}(\boldsymbol{\eta}) = [\gamma''\boldsymbol{H}_0 + 2\boldsymbol{u}_0 \cdot \nabla(\gamma'\boldsymbol{H}_0) - \nabla(\gamma'\boldsymbol{u}_0 \cdot \boldsymbol{H}_0)]\hat{\eta}.$$

Our governing equation for $\boldsymbol{\eta}$, (3.1), now simplifies to

$$(\ddot{\gamma} - \gamma'')\boldsymbol{H} + 2\boldsymbol{u}_0 \cdot \nabla[(\dot{\gamma} - \gamma')\boldsymbol{H}] = \nabla(\cdot). \tag{6.16}$$

Next, we expand the second term on the left, ignore spatial variations in $\boldsymbol{H}_0$ by comparison with those of γ, and eliminate the unknown gradient function by taking the curl of (6.16). The result is

$$\left(\frac{\partial}{\partial t} - \frac{\partial}{\partial \beta}\right)^2 \gamma' = 0 \tag{6.17}$$

which has general solution

$$\gamma' = \gamma_1(\beta + t) + t\gamma_2(\beta + t). \tag{6.18}$$

This represents two transverse waves of fixed shape propagating along the surface $D = \text{constant}$ in a direction normal to $\boldsymbol{H}_0$. The second wave, γ_2, grows linearly in time and is reminiscent of the algebraic instability found in non-Beltrami Euler flows. However, because we can only follow such waves for a limited period of time we cannot conclude that (6.19) represents an unstable mode, although it does seem plausible.

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